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COMPLEX ANALYSIS, IMAGINARY NUMBERS IN REAL WORLD - SOME IMPORTANT THEOREM AND THEIR PROOFS

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ABSTRACT

This article discusses some introductory ideas associated with complex number More complex the society, the more complex is the need of the mathematical needs. Here we look at the n^th roots and solution of the equations z^n = 1. Since last four centuries, the complex systems are being studied with increasing intensity as it is widely accepted as mathematical truth, for representation's sake or by choice. As more abstract proof evolved, it gave mathematicians confidence to continually develop the techniques to solve the complex system. Today, the system is actually a study of an independent subject called as complex analysis. The advanced studies of the Complex Numbers and the expansion and simplifications of proofs provided new and broad perspectives to approach the many branches of mathematics.

KEYWORDS: *Complex Numbers, Analytical Functions, Series, Sequence, Convergence, Residues*

INTRODUCTION

In the year, The Great Swiss Mathematician Leonard Euler.. Came with array of principles which became the foundation for the modern mathematics..He named a number *'i'* as Iota, square of which is−1. This Iota or *'i'* is defined as imaginary unit of a complex number. With the introduction of Iota *'i'*, we can interpret the square root of a negative number as a product of a real number with Iota- *i*...Therefore, we can denote it as x = ± i.

Any number expressed in the form of x + iy [where x & y are real numbers and i = √−1], is known as a complex number. A Complex Number is a combination of a Real Number and an Imaginary Number. A complex number is, generally, denoted by the letter z…i.e. z = x + iy, 'x' is called the real part of z and is written as Re (x+iy) and 'y' is called the imaginary part of z and is written as Im (x + iy). If x = 0 and y ≠ 0, then the complex number becomes iy which is a purely an imaginary complex number… –4i, 1/ 2i, 6i, 5i and πi are all examples of purely imaginary numbers.

Below is the illustration of the complex plane. The real part of a complex number z = x + iy is x, and its imaginary part is. We write C for the set of all complex numbers…C = { z : z= x + iy : x,y Є R},

Where $i = \sqrt{-1}$, *i.e* i^2 = 1, … $i^n \in \{-1, 1, i, 1\} \; \forall \; n \in z$..

Figure 2 shows If x = 0, the number is said to be Purely Imaginary, and if y = 0, the number is Real…Note that Zero is the only number which is both Real and Imaginary Number. X and y are respectively the real and Imaginary parts of the complex number x + iy, denoted by z i.e z= x + iy other standard notation, we use to denote a complex number is w

+ u+i The Real and Imaginary parts of a complex number are denoted by Re z, and Im z A complex Conjugate of a complex number z = x + iy is x-iy and is denoted by $\bar{z}$ = x – iy.

The Number is real when conjugate of z= $\bar{z}$....The Real and Imaginary parts of a complex number can be expressed in terms of the complex number and its conjugate:

Re z = $\frac{z+\bar{z}}{2}$, and Im z = $\frac{z-\bar{z}}{2i}$ ……

The Modulus of a Complex Number z= x + iy is given by

$$|z| = \sqrt{\sqrt{x^2+y^2}}$$

$|z|^2 = x^2 + y^2 = (Re\, z)^2 + (Im\, z)^2 \geq (Re\, z)^2$

In Geometric Interpretation, we use Polar Coordinates. If the polar coordinates of (x,y) are (r, θ), then x= r cos θ and y = r sin θ, Hence z = x + iy = (Cos θ + i Sin θ)..r ≥ 0, r = $|z| = \sqrt{x^2+y^2}$..The Polar angle φ, is called the is called Argument (or Amplitude) of the Complex Number.

Binomial Equation:--Now the n^{th} power of the Complex Number z = r (Cos θ + i Sin θ), is given by $z^n = r^n$ (Cos n θ – i Sin n θ).

The Formula is Trivially Valid for z= o, and since

$z^{-1} = r^{-1}$ (Cos θ – i Sin θ) = r^{-1} [Cos(-θ) + i Sin (-θ)], it holds also for when n is a negative integer

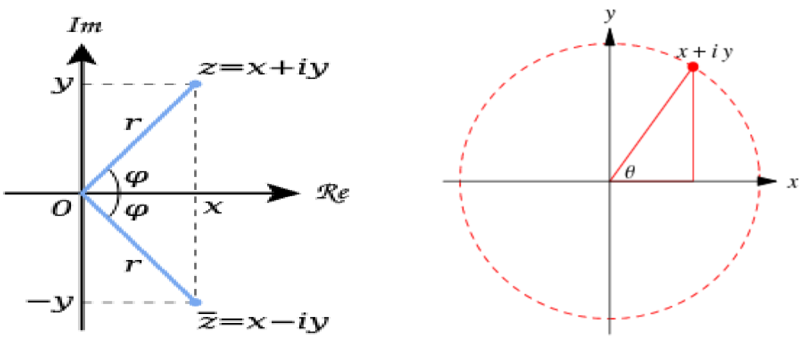

Figure 1

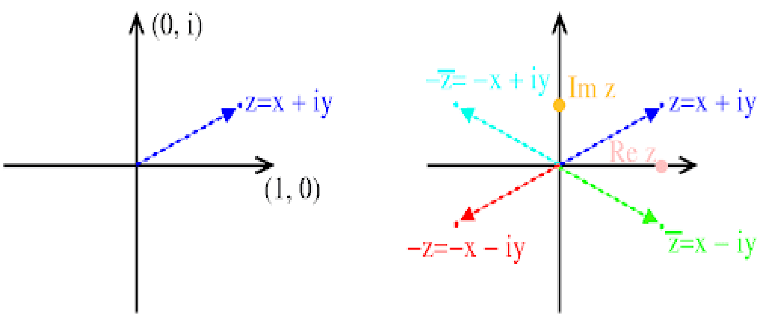

Figure 2

For r =1, we have De Movers Theorem

$(\cos\theta + i \sin\theta)^n$ = Cos n □ + i Sin n □

It may be noted that n^{th} root of any complex number z≠ 0, have same Modulus and their Arguments are Equally Spaced.

In the Geometry of a Complex Number, we know the equation of a Circle, with Center at "a" and Radius "r" is given by $|z-a|$ = r, z = $re^{i\theta}$ is the Parametric representation of a Circle with Radius "r"

An Inequality $|z-a|$ < r describes the Inside of the Circle

In Algebraic Form it can be Written as (z-a) ($\bar{z}$ -$\bar{a}$) = r^2

The Connection between $e^{i\varphi}$, Cos φ. and Sin φ is given by Euler's Theorem as

$e^{i\theta}$ = Cos □ + i Sin □

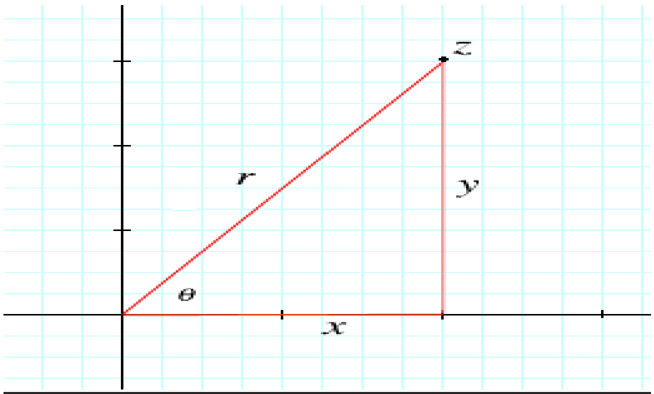

Figure 3

Note that the complex number cos θ + *i* sin θ has absolute value 1 since $\cos^2\theta + \sin^2\theta = 1$ for any angle θ. Thus, every complex number *z* is the product of a real number |z| and a complex number cos θ + *i* sin θ.

A Straight Line in a complex plane: is given by a Parametric Equation z= a+ bt, where a and b are complex numbers and b ≠ 0, "t", is the parameter. The parameter is a Real Value.

{Z = a + bt : t Є R}

Continuous Function: A function f(z) is continuous at a point at "a" if and only if

$\lim_{x\to a} f(x)$ = f(a),

If f(x) is continuous, then Re f(x), Im f(x) and $|f(x)|$ are also continuous

Derivative of a Complex Number: Let w = f(z)..If the limit is z→a, then the derivative f′ (z) is defined by

f′ (a) = $\lim_{z\to a}\frac{f(z)-f(a)}{z-a}$, Provided the limit exists. If the limit exists we say f is differentiable at "a" and also then only f is analytic at a.

Analytic Function: A function f(z) is an Analytic function, if it has a Complex Derivative f′ (z)

i.e f′ (z) = $\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{z \to 0} \frac{f(z) - f(z0)}{z - z0}$ --

The limit has to exist and be the same, no matter how you approach "z0 or just 0"

Partial Derivatives as Limits: Let us remind the Partial Differential equations before we proceed for the Cauchy Riemann's Equations:-

If u(x, y) is a function of two complex variables then the partial derivatives of u are defined as

$\frac{\partial u}{\partial x}$ (x,y) = $\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y)-u(x,y)}{\Delta x}$, when "y" is constant

& when "x" is constant then

$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y)-u(x,y)}{\Delta y}$

Series of a Complex Number

The summation of a series a1+ a2+a3+ a4 +…+ an +.., denoted $by:-$

$\sum_{n=1}^{\infty} an =$ a1+ a2+a3+ a4 +….+ an +…, is called an infinite series of a complex number.

Now we write sn = a1 +a2 +….+an, Here sn is called the nth partial sum of the series $\sum_{n=1}^{\infty} an$ ….

The Sequence {sn} is called the Sequence of partial sum of $\sum_{n=1}^{\infty} an$

A sequence is called a Cauchy Sequence, if it satisfiers the following condition:

If given any ε>0 ∃ n0: $|an - am|$ <ε, whenever n≥ n0, and also m≥ n0

A sequence is convergent if and only if it's a Cauchy sequence

Cauchy's Criterion for Uniform Convergence:-

$\sum_{n=1}^{\infty} \text{an}$ is convergent if and only if to each ε >0 ∃ n0 ∈N : | an+1 + an+2 +….+an+p | <ε ∀ n ≥ n0, p >1

The Sequence {fn (z)} converges uniformly on E, if and only if to each ε >0, there exists an n0 such that | fm – fn | <ε ∀ m,n ≥ n0 and all Z Є E…

Proof: Suppose {fn (z)} converges uniformly on f(z) say on E. Let ε>0 be given…. ∃ n0 ∈N, Such that |fn(z) – fm(z)| = |fm(z) – f(z) + f(z) – fn(z)| ≤ | fn(z)- f(z) | + |fm(z) – f(z)| <$\frac{\epsilon}{2}$ +$\frac{\epsilon}{2}$ = ∈

|fm(z) – fn(z)| <∈ ∀ m,n ≥ n0 ∀ z ∈ E…Conversely suppose for ∈ >0 ∃ n0 such that

|fm(z) – fn(z)| < ∈ ∀ m,n ≥ n0 ∀ z ∈ E

⟹{ fn(z)} is a Cauchy Sequence ⇒ { fn(z)}is convergent and the convergence is uniform as n0 depends on ∈.Hence Proved

Power Series: A formal sum of the form $a_0 + a_1 + a_2 + b\ldots\ldots+a_n\ldots.$, where the coefficient a_n and the variable z are complex Numbers, is called a Power Series:- Denoted as

$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 + a_2 + \ldots..+a_n +$

$\sum_{n=0}^{\infty} a_n (z- z_0)^n$ is called the Powered Series w.r.to Center z_0.

For every Power Series $\sum_{n=0}^{\infty} a_n \ z^n$ there exists a number R : $0 \leq R \leq \infty$,. called the Radius of Convergence, with the following properties

- The series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for every z with $|z| <R$, If $0 \leq \rho \leq R$, the convergence is uniform for $|z| \leq \rho$
- If $|z| >R$, the terms of the series $\sum_{n=0}^{\infty} a_n z^n$ get bounded and hence the series is Divergent
- If $|z| >R$, the sum of the series is an analytic function. Then Derivative can be obtained by term wise differentiation and the derived series has the same Radius of Convergence

This is called **Abel's Theorem** which gives the information about the circle of convergence of a Power series.

Now if $\sum_{n=0}^{\infty} a_n$ converges then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ the ends to f(1) as z approaches 1 in such a way that $\frac{|1-z|}{(1-|z|)}$ Remains bounded. This is called Abel's Limit Theorem for power series that converges at a Point on the circle of Convergence

Cauchy Riemann's Equations

Theorem: If w= f(z)is a Complex valued Analytic Function then :-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x} \qquad (1)$$

The Eqns (1) are called Cauchy- Riemann's Equations

Proof: If W= F (z) is a C.V Analytic Function & f '(z) exists, whenever f(z) is defined

i.e $f'(z) = \frac{\lim_{h\to 0} f(z+h) - f(z)}{h}$ exists.

In other words the Quotient $\frac{f(z+h) - f(z)}{h}$ should approach the same limit regardless of the way in which h approaches zero

Let h→0 through Real Values, then :- $\frac{f(z+h) - f(z)}{h} =$

$$= \left[\frac{u(x+h,)-u(x,y)}{h}\right] + i \left[\frac{[v(x+h,y)- v(x,y)]}{h}\right]$$

i.e $\quad f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} =$

$= Ux - i Vx \qquad (2)$

Now let h→0 through Imaginary Values:- Say h = ik, k ∈ R, then-

f '(z) = $\text{Lim}_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{[u(x,y+k)-u(x,y)]}{ik} + i\lim_{k\to 0}\left[\frac{v(x,y+k)-v(x,y)}{ik}\right]$

$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$= -1\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

= i Uy + Vy (3)

Comparing Equations (2) & (3) we get

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \& \quad \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\}$$ Equations (4) C. R Equations

The real and imaginary parts should satisfy C.R Equations…

Note: From the above Equations, we have several Formulas for f '(z), namely:

f '(z) = Ux + i Vx = - i Uy + VY = Ux + i (-Uy)

= Ux - i Uy ∵ Uy =-Vx

= - i (- Vx) + Vy = i Vx + Vy ∵ Uy =-Vx

Moreover,

$$|f'|^2 = \left\{\frac{\partial u}{\partial x}\right\}^2 + \left\{\frac{\partial u}{\partial y}\right\}^2$$

$$= \left\{\frac{\partial u}{\partial x}\right\}^2 + \left\{\frac{\partial v}{\partial y}\right\}^2 = \frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}$$

$= J\left[\frac{u,v}{x,y}\right]$ The Jecobion of u,v w.r.to x,y.

Now, The Derivative of an Analytical function is again an Analytical function. This implies that u,v will have continuous partial Derivatives of all orders.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$

$\Longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

A function u(x,y) & v(x,y) which satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ is called a Harmonic function

Cauchy's Integral Theorem / Cauchy's Theorem for an Analytic Function

Theorem:-If f (z) is a Complex Valued Analytical function in a Region ℂ, i.e f '(z) is continuous at all points, then

$$\oint_\gamma f(z) = 0 \quad (1)$$

For every cycle γ which is homologous to Zero in ℂ.

Proof: Consider the figure:- Here we have a closed curve γ, an analytic function f (z), within and on a closed curve γ, contained in a region ℂ

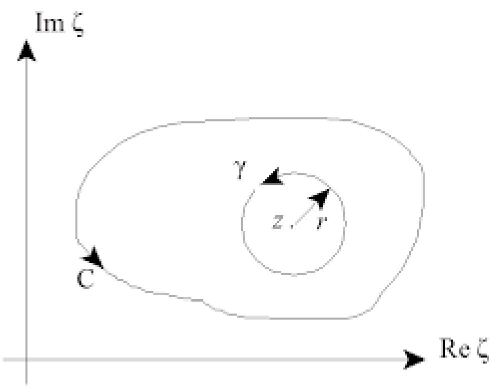

Figure 4

We have z = x + iy (2)

dz = dx + idy (3)

Now f (z) = u + (4)

Now Integrating f (z) in a closed curve γ, we get:

$$\oint_\gamma f(z)\mathrm{dz} = \oint_\gamma (u+iv)(dx+idy) \quad (5)$$

$$\oint_\gamma f(z)\mathrm{dz} = [udx + iudy + ivdx - vdy] \quad (6)$$

Now separating the Real & Imaginary Parts we get:

$$\oint_\gamma f(z)\mathrm{dz} = \oint_\gamma (udx - vdy) + i \oint_\gamma (vdx + udy) \quad (7)$$

Now from Green's Theorem we have:

$\oint_\gamma Pdx + Qdy$ = c$\int\int\{\frac{\partial Q}{\partial X}$ - $\frac{\partial P}{\partial Y}\}$ dx dy

Applying Green's Theorem in Equation in Eqn (7), we get

$\oint_\gamma udx - vdy = \int_C \int \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$ dx dy

= $- \int_C \int \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dx\, dy$

& $\oint_\gamma vdx + udy$ = $\int_C \int \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right]$ dx dy

$\oint_\gamma f(z)\mathrm{dz}$ = $- \int_C \int \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dx\, dy$ + i $\int_C \int \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right]$ dx dy (8)

Now we know that f (z) is an analytic function completely contained in Region/Domain C, and its Real and Imaginaray parts Satisfy C.R Equations i.e $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Now putting $\frac{\partial u}{\partial x}$ as $\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}$ as $-\frac{\partial v}{\partial x}$ in Eqn (8), the Terms of the Real and Imaginary Parts in the RHS of the Equation cancels each other and Vanishes Independently & becomes Zero, showing That

$\oint_\gamma f(z)$ = 0 Hence Proved

Cauchy's Integral Formula

Theorem: Suppose that f (z) is Analytic in an open Disk Δ, and let γ be a closed curve contained in Δ … Let "a" be a point inside γ, then :

n(γ,a) f (a) = $\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}$ dz, where n(γ,a) is the index of "a" w.r.to γ… (1)

Proof

Consider the Figure 5 We have an open disk Δ, a closed curve γ, and a point "a" inside γ…. Since "a" is ma point within γ, we shall enclose it by a closed curve/ circle γ_1 center at "a" and radius "r"…, such that γ_1 is entirely within γ..

r→0,

r>0

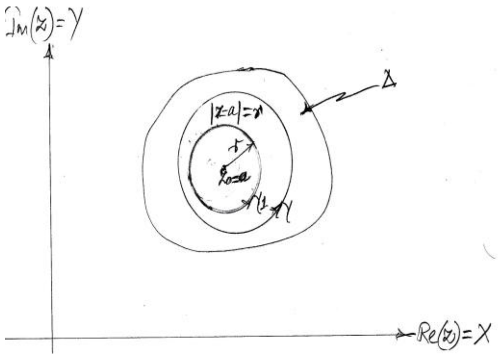

Figure 5

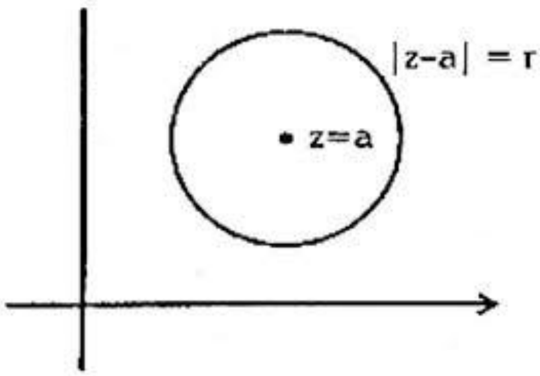

Figure 6

The function $\frac{f(z)}{z-a}$ is analytic within and on the boundaries of the annular region between γ & γ_1 (Except on the point "a") in Δ.

Now as a Consequence of Cauchy's Integral Theorem for an Analytic Function, we get

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = \oint_{\gamma 1} \frac{f(z)}{z-a} \tag{2}$$

Now the circle γ_1 (Circle with center at "a" and radius "r"), can be written

as:- $|z - a| = r \quad \Rightarrow z - a = re^{i\theta}$

Or $\quad z = a + re^{i\theta} \quad \Rightarrow dz = ire^{i\theta}\, d\theta,\ 0 \le \theta < 2\pi,$

Using the results in the RHS of the Equation (2), we get:-

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = \oint_{\theta=0}^{2\pi} \frac{(a+re^{i\theta})}{re^{i\theta}} . ire^{i\theta}\, d\theta, \quad r > 0$$

$r \to 0$

$$\Rightarrow \oint_{\gamma} \frac{f(z)}{z-a} dz = i \oint_{0}^{2\pi} f(a) d\theta$$

$$\Rightarrow \oint_{\gamma} \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$$

$$\Rightarrow f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a}\, dz \tag{3}$$

If multiple loops are made around the point "a", then the above Equation(3) becomes

$$n(\gamma,a).\, f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a}\, dz \tag{4}$$

Where n(γ,a). Is the index of "a", w.r.to γ…The above equation (4) is called Representation Formula…

In fact "a" can be replaced by any $\xi \in \Delta - \gamma$, so that the above formula becomes:

$$f(\square) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-\xi}\, dz$$

This is called the Cauchy's Integral Formula Proved

Cauchy's Residue Theorem

If f(z) is analytic except for isolated singularizes a_j in a region Ω, then for any Cycle γ which is homologous to Zero in Ω and that does not pass through any a_j, we have

$$\frac{1}{2\pi i} \oint_{\gamma} f(z)dz = \sum_{j=0}^{m} n(\gamma, a)\, \mathrm{Res}_{z=aj} f(z)$$

Proof: First we assume that the Isolated Singularities are finite in numbers and they are $a_1, a_2, a_3, \ldots, a_m$ in a region Ω.

Let $\Omega^I = \Omega - \{a1, a2, a3, - - - - - - - - - - -, am\}$

To each a_j there is a $\delta_j > 0$, such that $0 < |z - aj| < \delta_j$ lies in Ω^I

If $\xi_j = \{z: ||z - aj| |rj\}$ where $0 < r_j < \delta_j$

Then by the following Theorem { If f(z) has an isolated singularity as "a" then there is an unique complex number R, such that $f(z) - \frac{R}{z-a}$ is the derivative of a single valued analytical function in the annulus $0 < |z - a| < \delta$ } we get:-

$$\mathrm{Res}_{z=aj} f(z) = \frac{1}{2\pi i} \oint_{cj} f(z)dz \tag{1}$$

Also since $\gamma \sim \sum_{j=1}^{m} n(\gamma, a)\, c_j$ with respect to Ω^I

Therefore

$$\oint_{\gamma} f(z)dz = \int_{\sum_{j=1} n(\gamma,a)}^{m} f(z)dz = \sum_{j=1}^{m} n(\gamma, a) \int_{cj} f(z)dz$$

$$\int_{\gamma} f(z)dz = \int_{\sum_{j=1}^{m} n(\gamma,a) cj} f(z)dz = \sum_{j=1}^{m} n(\gamma, aj) \int_{cj} f(z)dz$$

Which can be written, in view of Equation (1)

$\int_\gamma f(z)dz = \sum_{j=1}^{m} n(\gamma, a)$. 2πi Res $_{z=aj}$ f(z)

i.e $\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{j=1}^{m} n(\gamma, a)$ Res $_{z=aj}$ f(z)

Some important notes on Residue of a Complex Number

- Residue at (z=a) = $\lim_{z-a}(z-a)\ f(z)$
- Residue at ∞ = $\lim_{z\to\infty}[-z\,f(z)]$
- Residue of pole of order m =

$$\frac{1}{(m-1)!}\lim_{z\to z0}\frac{d^{m-1}}{dz^{m-1}}\{(z-z0^m)f(z)\}$$

Morera's Theorem

Suppose f(z) is continuous in a region Ω and $\int_\gamma f(z)dz = 0$ ∀ closed curves γ in Ω, then f(z) is analytic in Ω…

Proof: Since f(z) is continuous in Ω, and $\int_\gamma f(z)dz = 0$ ∀ closed curves γ, we have that f(z) dz is an exact differential

Hence there is a continuous function F(z) on Ω, such that F^I(z)= f(z) ∀ z ∈ Ω

That is, F (z) is analytic on Ω

From the Theorem {If f (z) is an analytic function defined on a region Ω then f(z) has derivative of all orders}

Now we see that F^I (z) is also Analytic

That is f(z) is analytic on Ω. Proving the Theorem Hence proved

Liouville's Theorem

A function f(z) which is analytic in the entire finite complex plane and is bounded, is a constant function..

Proof: Let $|f(\xi)|$ ≤ M ∀ ξ ∈ C

Fix a ∈ C

Let γ: $|\xi - a|$ = R

Then f^I(a) = $\frac{1}{2\pi I}\int_\gamma \frac{f(\xi)}{(\xi-a)^2}$ dξ

$|f^I(a)| \le \frac{1}{2\pi i}\int_\gamma \frac{|f(\xi)|}{|\xi-a|^2}|d\xi| \le \frac{1}{2\pi}\frac{M}{R^2}\int_\gamma |d\xi| = \frac{1}{2\pi}\frac{M}{R^2}$. 2πR = $\frac{M}{R}$

Now letting R⇢∞, we get that

$|f^I\ (a)|$ = 0

i.e $f^{I}(a) = 0$

Since "a" is arbitrary f(z) is a constant function

Polynimials

Def: **(1)** If P(z) & Q(z) are the two Polynimials then

R (z) = $\frac{P(z)}{Q(z)}$ is a Rational Function We assume P(z) & Q(z) have no common factor and hence no common Zeros.

If $\propto$ is Zero of Q(z), we Define R(z) = ∞, We can consider R(z) as a function with the values in the extended complex plane and as such it is continuous

The Zeros of Q (z) are called Poles of R(z)

The order of Pole of R (z) is defined to be the order of the corresponding Zeros of Q(z)

A pole of order one is called a Simple Pole

All Zeros of R (z) are given by the Zeros of P(z)

Important

- Zeros of R(z) are given by Roots of P(z)
- Zeros of Q(z) are called Poles of R(z)
- Poles of R(z) are the Zeros of Q(z)

Luca's Theorem

If all zeros of a polynomial P(z) lie in a half plane, then all Zeros of the Derivative $P^{I}(z)$ lie in the same half plane

Proof: Let $P(z) = a_0 + a_1 z + a_2 z^2 + \quad + a_n z^n$ be a Polynomial of degree n>0, with the zeros, say $\propto_1, \propto_2, \propto_3, \quad , \propto_n$..

Then we can write P(z) as:

$P(z) = a_n (z-\propto_1)(z-\propto_2)\ldots\ldots(z-\propto_n)$

Formal Logarithmic Differential leads to:

$$\frac{P^{I}(z)}{P(z)} = \frac{1}{z-\propto 1} + \frac{1}{z-\propto 2} + \qquad + \frac{1}{z-\propto n}$$

$$= \sum_{k=1}^{n} \frac{1}{z-\propto k}$$

Now Let H= $\left\{z: Im\left[\frac{z-a}{b}\right] < 0\right\}$ be the Right help plane that contains all the Zeros of P(z), namely $\propto_1, \propto_2, \propto_3, \propto_n$

Note that H is determined by the line

Z= a+bt

Since $\propto_k$'s lies in H, we have

$I_m \left[\frac{\propto k - a}{b}\right] < 0$, k =1,2,3,4,….., n.. (1)

& $- I_m \left[\frac{\propto k - a}{b}\right] > 0$, k=1,2,3,4,….., n

We prove the theorem by Showing That any $z_0 \notin H$ can never be a Zero of P(z), so that all the Zeros of $P^I(z)$, if any, should lie within H.

So let $z_0 \notin H$.

$I_m \left[\frac{z0 - a}{b}\right] \geq 0$ (2)

Now $\frac{z - \propto k}{b} = \frac{z0 - a + a - \propto k}{b} = \frac{z0 - a}{b} - \frac{\propto k - a}{b}$

$\mathrm{Im}\left[\frac{z0 - \alpha k}{b}\right] = \mathrm{Im}\left[\frac{z0 - a}{b}\right] - \mathrm{Im}\left[\frac{\alpha k - a}{b}\right] > 0.,$ from (1) & (2)

Hence $\mathrm{Im}\left[\frac{b}{z0 - \alpha k}\right] < 0$

Now $\frac{P^I(Z)}{P(z)} = \sum_{k=1}^{n} \frac{1}{z - \propto k}$

$\frac{bP^I(z0)}{P(Z0)} = \sum_{k=1}^{n} \frac{b}{z - \alpha k} < 0$

$\frac{bP^I(z0)}{P(Z0)} \neq 0.$ $\therefore$ $P'(z0) \neq 0$

Thus $z_0 \notin H$ implies that z_0 is not a Zero of $P'(z)$…

Hence all the Zeros of $P'(z)$ also lie in H.

Hence Proved

REFERENCES

1. *Complex Analysis by - Lars V Ahlfors.*
2. *Complex Analysis by Elias. M. Stein & Rami Shakarchi.*
3. *Complex Analysis from Bak and Newman (Springer).*
4. *www.mathwarehouse.com.*